Compensation, Incentives, and the Duality of Riskiness and Risk Aversion

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by

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The Common Folklore

Giving executives options makes them more willing to take risks. DeFusco, et.al. (1990), "The asymmetric payoffs of call options make it more attractive for managers to undertake risky projects."

Increasing volatility makes an option more valuable (see, e.g., Merton (1976), Cox and Ross (1976), Haugen, et. al.(1981), Smith, et. al.(1982), and Smith, et. al. (1985)).

Utility theory: Lambert, et. al. (1991) describe the sensitivity of the agent's valuation of compensation to variables such as wealth and the degree of risk aversion.

In a model where a portfolio manager adjusts to a convex incentive structure, Carpenter (2000) observes that the manager may behave counterintuitively. Lewellen (2001) makes a similar observation for executives.

But the general question of why and under what conditions this might occur remains somewhat mysterious.

Agency theory (see, e.g., Ross (1973) or Holmstrom (1979) explores optimality, but, unfortunately, such efforts have crowded out the study of the behavior of the agent given the specific contract forms of the sort that are commonly observed in practice.

I. Some Simple Examples: Put and Call Option Fee Schedules

Consider an executive whose compensation consists of a fixed fee together with some options on the company's stock. We will assume, as is usual, that the options are not fungible and that the executive must hold them to maturity.

Example 1:

$$f(x) = \max\{x - a, 0\}$$

$$\begin{aligned} U(f(x)) &= U(\max\{x - a, 0\}) \\ &= U(x - a), x \geq a \\ &= U(0), x \leq a \end{aligned}$$

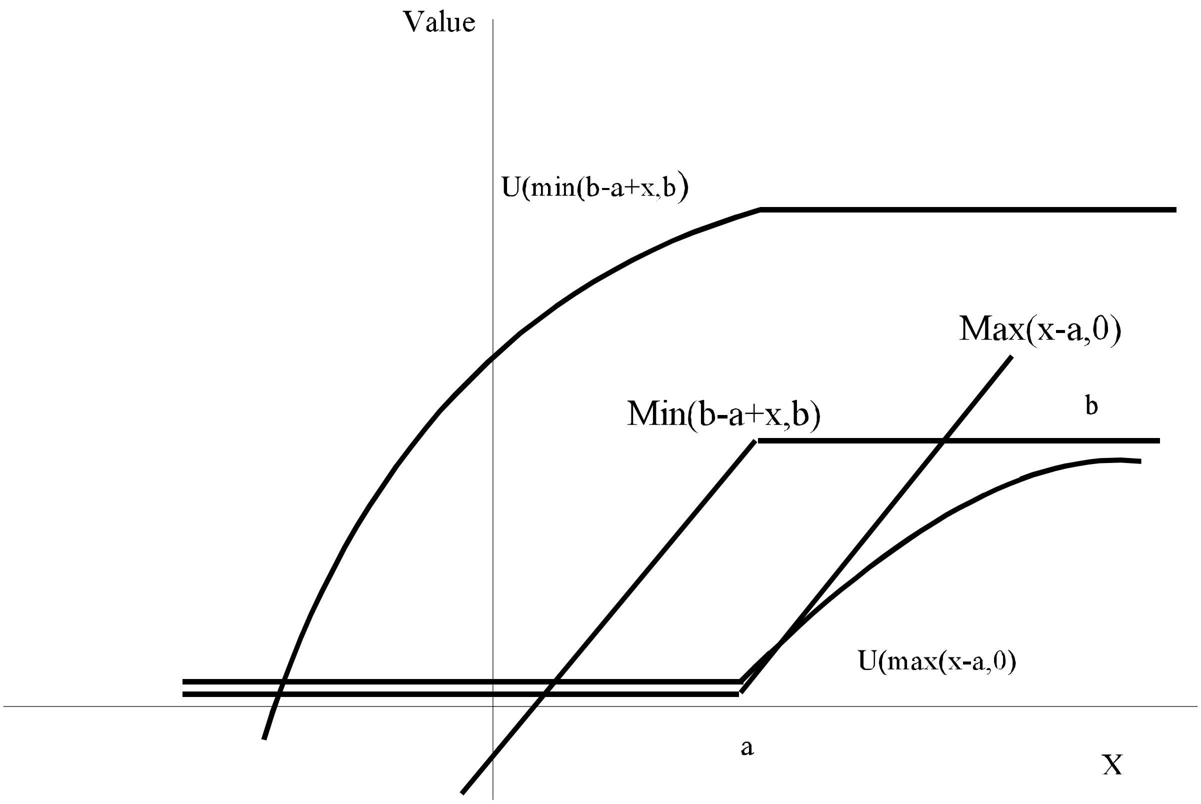
Example 2:

$$f(x) = b - \max\{a - x, 0\} = \min\{b - a + x, b\}$$

$$\begin{aligned} &U(f(x)) = U(\min\{b - a + x, b\}) \\ &= U(b - a + x), x < a \\ &= U(b), x \geq a \end{aligned}$$

Figure 1

The Utility of Put and Call Incentive Schedules



II. The General Theory

Definition 2.1

A fee schedule, f, concavifies a utility function, U, iff there exists a monotone concave function, G, such that

$$U(f) = G(U)$$

Definition 2.2

A fee schedule, f, convexifies a utility function, U, iff there exists a monotone convex function, G, such that

$$U(f) = G(U)$$

Since G^{-1} is concave iff G is convex, f convexifies U iff there exists a concave H such that

$$U = H(U(f))$$

We will let A to denote the coefficient of absolute risk aversion,

$$A = -\frac{U''(x)}{U'(x)}$$

Since U is monotone, for any f there always exists a function G such that,

$$U(f) = G(U)$$

Furthermore, f is monotone if and only if G is monotone, since

$$U'(f)f' = G'(U)U'$$

.

Differentiating again we have,

$$U''(f)(f')^2 + U'(f)f'' \\ = G''(u)(U')^2 + G'(u)U''$$

which rearranges to

$$G''(u)(U')^2 = U'(f)f'[A + \frac{f''}{f'} - A(f)f']$$

Hence, we have the following result.

Theorem 2.1

The compensation schedule, f, concavifies (convexifies) U iff

$$\frac{f''}{f'} \leq (\geq) A(f) f' - A$$

Proof: See above.

□

The following lemma verifies what is certainly true about the folk result.

Corollary 2.1

The compensation schedule, f, concavifies (convexifies) U for all U only if f is concave (convex).

Proof:

Taking U to be risk neutral, A = 0, then the result follows from Theorem 2.1 and the monotonicity of f.

□

Indeed, concavity of f is necessary for U(f) to even be risk averse for all concave U. However, while concavity (convexity) of f is necessary for the derived utility function to be concavified (convexified), it is not sufficient. To see this, observe that for any point x such that $f(x) \neq x$ and for any values $f'(x)$ and $f''(x)$, we can set A(x) sufficiently large while holding A(f(x)) fixed and construct a violation of the condition of Theorem 2.1 for f to concavify U.

This verifies the following important corollary.

Corollary 2.2

There is no compensation schedule that concavifies (convexifies) all U.

Proof: See above argument.

□

Theorem 2.2

The compensation schedule, f, concavifies all $U \in$ DARA iff f is concave, $f \leq x$, and $f' \geq 1$, and it convexifies all $U \in$ DARA iff f is convex, $f \geq x$, and $f' \leq 1$. The compensation schedule f concavifies all $U \in$ IARA iff f is concave, $f \geq x$, and $f' \geq 1$, and f convexifies all $U \in$ IARA iff f is convex, $f \leq x$, and $f' \leq 1$.

Proof:

Suppose $U \in$ DARA. If $f \leq x$, then $A(f) \geq A$, and if $f' \geq 1$, then $A(f) f' - A \geq 0$. Since, by concavity, $f'' \leq 0$, we have

$$\frac{f''}{f'} \leq A(f)f' - A$$

and, by Theorem 2.1, f concavifies U. To prove necessity observe that if A is constant, then we must have,

$$A(f)f' - A = A[f' - 1] \geq \frac{f''}{f'}$$

Picking U risk neutral, i.e., $A = 0$, verifies that $f'' \leq 0$. Picking A arbitrarily large reverses the inequality unless $f' \geq 1$. Now, suppose that $f(x) > x$ for some x. We can set $A(f)f' - A$ arbitrarily small by setting $A(x)$ as large as desired relative to A(f), and this also reverses the inequality. The proofs for convexity and for IARA are similar.

□

Definition 2.3

$$A(dc) \equiv \{f \mid f \text{ is concave, } f \leq x \text{, and } f' \geq 1\}$$

$$A(dx) \equiv \{f \mid f \text{ is convex, } f \geq x \text{, and } f' \leq 1\}$$

$$A(ic) \equiv \{f \mid f \text{ is concave, } f \geq x \text{, and } f' \geq 1\}$$

$$A(ix) \equiv \{f \mid f \text{ is convex, } f \leq x \text{, and } f' \leq 1\}$$

The next theorem shows that if a utility function is concavified or convexified for all of the members of one of the classes defined above, then it must be DARA or IARA depending on which class is used.

Theorem 2.3

If $f \in A(dc)$ or $f \in A(ic)$ implies that f concavifies U, then U is DARA or IARA, respectively. If $f \in A(dx)$ or $f \in A(ix)$ implies that f convexifies U, then U is DARA or IARA, respectively.

Proof:

The proofs are all similar so we will only do the first. Assume, then, that $f \in A(dc)$ implies that f concavifies U. From Theorem 2.1 a necessary condition for f to concavify U is that

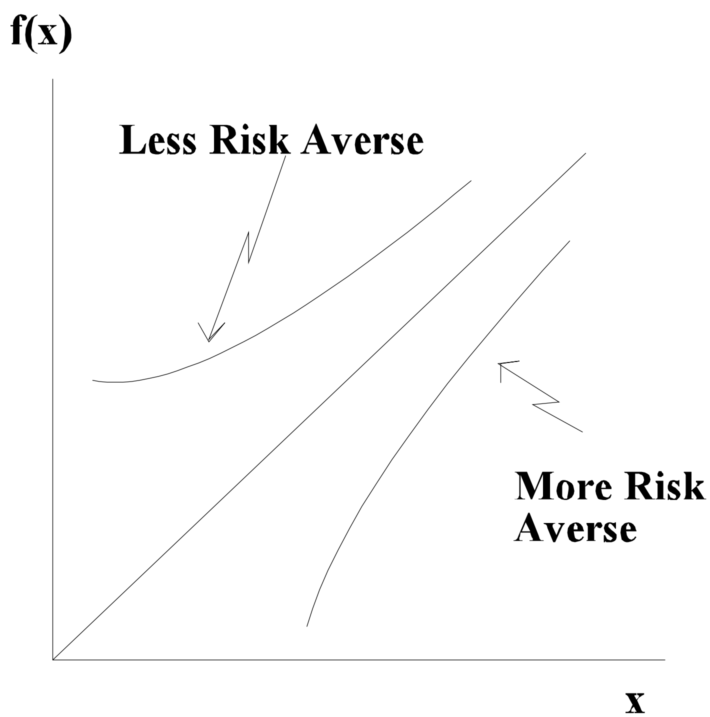
$$\frac{f''}{f'} \leq A(f)f' - A$$

If $A(f)f' - A < 0$, then for $f'' < 0$ sufficiently close to 0 we can violate this condition for some $f \in A(dc)$, hence we must have $A(f)f' - A \geq 0$ for all $f \in A(dc)$. Since, aside from concavity, the only other conditions on f are $f \leq x$, and $f' \geq 1$ we must also have $A(f) \geq A(x)$ for all $f \leq x$. Hence, U must belong to DARA.

□

Figure 2

Risk Inducing and Risk Averting Fee Schedules for DARA Utility



III. Risk Inducing and Risk Averting Modifications of an Existing Schedule

Transform an existing schedule, f(·), to g(f(·)):

Whether this concavifies or convexifies the derived utility function, U(f(·)), depends on the shape of the transformation, H, defined by

$$U(g(f(x))) = H(U(f(x)))$$

Letting

$$z = f(x)$$

this becomes

$$U(g(z)) = H(U(z))$$

Corollary 3.1

The transform g(f(·)), concavifies the derived utility function for all U ∈ DARA iff g is concave, g ≤ x, and g′ ≥ 1, and it convexifies it for all U ∈ DARA iff g is convex, g ≥ x, and g′ ≤ 1. The alteration g concavifies the derived utility function for all U ∈ IARA iff g is concave, g ≥ x, and g′ ≥ 1, and g convexifies it for all U ∈ IARA iff g is convex, g ≤ x, and g′ ≤ 1.

Proof:

As described above, simply set $z = f(x)$ and apply Theorem 2.2.

□

Add options to an existing fee schedule:

$$U(f(x)+g(x)) = H(U(f(x)))$$

Letting

$$z = f(x)$$

and assuming f is strictly monotone, this becomes

$$U(z+g(f^{-1}(z))) = H(U(z))$$

Letting

$$q(z) = z+g(f^{-1}(z))$$

we have

$$U(q(z)) = H(U(z))$$

Corollary 3.2

Let A_g and A_f denote the coefficients of absolute risk aversion for g and f respectively. Adding g to an existing fee schedule, f, concavifies the derived utility function for all $U \in$ DARA iff $A_f \leq A_g$, $g \leq 0$, and $g' \geq 0$, and it convexifies it for all $U \in$ DARA iff $A_f \leq A_g$, $g \geq 0$, and $g' \leq 0$. The alteration g concavifies derived utility for all U $\in$ IARA iff $A_f \leq A_g$, $g \geq 0$, and $g' \geq 0$ and g convexifies it for all $U \in$ IARA iff $A_f \leq A_g$, $g \leq 0$, and $g' \leq 0$.

Proof:

Assuming that f is invertible, applying the transform,

$$q(z) = z + g(f^{-1}(z))$$

and differentiating we obtain

$$q'(f(x)) = 1 + \frac{g'(x)}{f'(x)}$$

where

$$x = f^{-1}(z)$$

and

$$q''(x) = \frac{f'(x)g''(x) - g'(x)f''(x)}{(f'(x))^3}$$

$$= \frac{g'(x)}{f'(x)^2}[A_f - A_g]$$

The result now follows by applying Corollary 2.3. (If f is not strictly monotone, let s(x) be any strictly monotone function with bounded derivatives and carry out the above analysis for f(x) + δs(x). Since U(f + δs + g) is more (or less) risk averse than U(f + δs) for all δ > 0, this must also hold for δ = 0.)

□

An interesting special case of this occurs when f(x) = x and we are simply adding to the entire payoff. Since $A_f = 0$, the conditions, for example, for convexifying a utility function with decreasing absolute risk aversion is that the addition g is nonnegative, convex and has a nonpositive slope. In other words, adding a positive, monotonely declining convex function will convexify an agent with decreasing absolute risk aversion.

Notice, then, that adding a call option will not convexify an agent with decreasing absolute risk aversion, but adding a put will. This, in turn, implies that to make agents more willing to take risks there should be more of a focus on offering downside protection than on offering them upside potential.

IV. The Convexity, Translation and Magnification Effects

Defining the derived utility function,

$$V(x) = U(f(x))$$

from the basic relation of Theorem 2.1 we have that

$$
\begin{aligned}
&A_V(x) - A(x) \\
&= -\frac{U''(f)f'}{U'(f)} - [-\frac{U''(x)}{U'(x)}] + [-\frac{f''}{f'}] \\
&= A(f)f' - A(x) + A_f(x) \\
&= [A(f) - A(x)] + A(f)[f' - 1] + A_f(x)
\end{aligned}
$$

The Decomposition Theorem

$$A_V(x) - A(x) = Translation\ Effect$$
$$+\ Magnification\ Effect$$
$$+\ Convexity\ Effect$$

$$Translation\ Effect = A(f) - A(x)$$

$$Magnification\ Effect = A(f)[f' - 1]$$

$$Convexity\ Effect = A_f(x)$$

Example:

$$U(x) = -e^{-Ax}$$

and

$$f(x) = \lambda g(x)$$

where g(x) is a positive, monotone, and concave function.

$$Translation\ Effect = 0$$

$$Convexity\ Effect = A_f(x) = -\frac{g''}{g'}$$

which is independent of λ.

$$Magnification\ Effect = A(f)[f' - 1]$$
$$= A[\lambda g' - 1]$$

Hence, if the risk aversion of the fee schedule,

$$A_f(x) = -\frac{g''}{g'} < A$$

then for λ sufficiently close to 0 the magnification effect will dominate the convexity effect and the derived utility function will be less risk averse than the original utility function. On the other hand, if

$$A_f(x) = -\frac{g''}{g'} \geq A$$

then even if $\lambda = 0$ the derived utility function will be no less risk averse than the original.

Conversely, if g is convex, then setting λ sufficiently high will make the derived utility function more risk averse than the original.

Example:

$$f(x) = x$$

All three effects are 0.

But, since

$$f(x) = a + bx$$

has no convexity effect it allows us to see the pure impact of the translation and magnification effects:

$$Convexity\ Effect = 0$$

$$Translation\ Effect = A(f) - A(x)$$
$$= A(a + bx) - A(x)$$

$$Magnification\ Effect = A(f)[f' - 1]$$
$$= A(a + bx)[b - 1]$$

V. Duality

Usually we say that a random variable y is less desirable than a random variable x iff for all U monotone and concave,

$$E[U(y)] \leq E[U(x)]$$

By contrast, if we let

$$y = f(x)$$

then

$$E[U(y)] = E[U(f(x)] \leq E[U(x)]$$

for all random x iff

$$f(x) \leq x$$

Definition 4.1

A random payoff y is said to be S-riskier by c (a constant) than a payoff x iff x is rejected for some U $\in$ S, and, whenever x is rejected by U $\in$ S, U prefers c to y, i.e.,

$$E[U(x)] \leq U(0) \Rightarrow$$
$$E[U(y)] \leq U(c)$$

This is a slight generalization of the usual definition that allows the reference origin of comparison for y to be translated by a constant c. Definition 4.1 allows us to state a more useful duality concept.

Definition 4.2

A function f is a risk inducing transform if for any random x, f(x) is S riskier by f(0) than x.

Theorem 4.1

A function f is an S risk inducing transform iff it concavifies $U \in S$.

Proof:

If f is risk inducing, then, for all x

$$E[U(x)] \le U(0) \Rightarrow$$
$$E[U(f(x))] \le U(f(0))$$

which is simply a statement that U(f(x)) is more concave than U(x). Conversely, if f concavifies U, then there exists a monotone concave function G such that

$$U(f(x)) = G(U(x))$$

which implies that if x is rejected by U, then

$$
\begin{aligned}
&E[U(f(x))] = E[G(U(x))] \\
&\leq G(E[U(x)]) \leq G(U(0)) \\
&= U(f(0))
\end{aligned}
$$

the condition for f being risk inducing.

□

From Corollary 2.2, though, we know that f cannot be a risk inducing transform for all $U \in MC$, and we must restrict the class of admissible utility functions, S. Theorem 2.2 provides a straightforward corollary. If we restrict S to the class of DARA utility functions, then it is immediate that f is DARA risk inducing iff $f \in A(dc)$. For the sake of completeness we offer the following formal statement. A parallel treatment for f being convexifying is equally straightforward.

Theorem 4.2

The compensation schedule, f, is DARA risk inducing iff f is concave, $f \le x$, and $f' \ge 1$, and it is IARA risk inducing iff f is concave, $f \ge x$, and $f' \ge 1$.

Proof:

Follows immediately from Theorems 2.2 and 4.1.

□