CENTRE NATIONAL DE LA RECHERCHE SCIENTIFIQUE

ÉCOLE POLYTECHNIQUE

Collateralized debt obligations (CDOs): modeling, pricing and risk management

Rama CONT & Nicole EL KAROUI

Centre de Mathématiques Appliquees

Ecole Polytechnique

Outline

- From securitization to CDOs
- The structure of a static CDO
- The market standard: the Gaussian copula model
- Efficient algorithms for CDO pricing in the Gaussian model
- Trading correlation: credit indices
- Beyond the Gaussian copula model: extracting implied default rates from market CDO quotes
- Dynamic trading strategies: portfolio insurance and credit CPPI
- Risk analysis of a CPPI strategy
- Conclusion

En collaboration avec/ In collaboration with

- ✓ Ying JIAO (Ecole Polytechnique & ENSIMAG)
- ✓ David KURTZ
- ✓ Andreea MINCA (Ecole Polytechnique)
- ✓ Ioana SAVESCU (Ecole Polytechnique)
- ✓ Peter TANKOV (Universite de Paris VII)

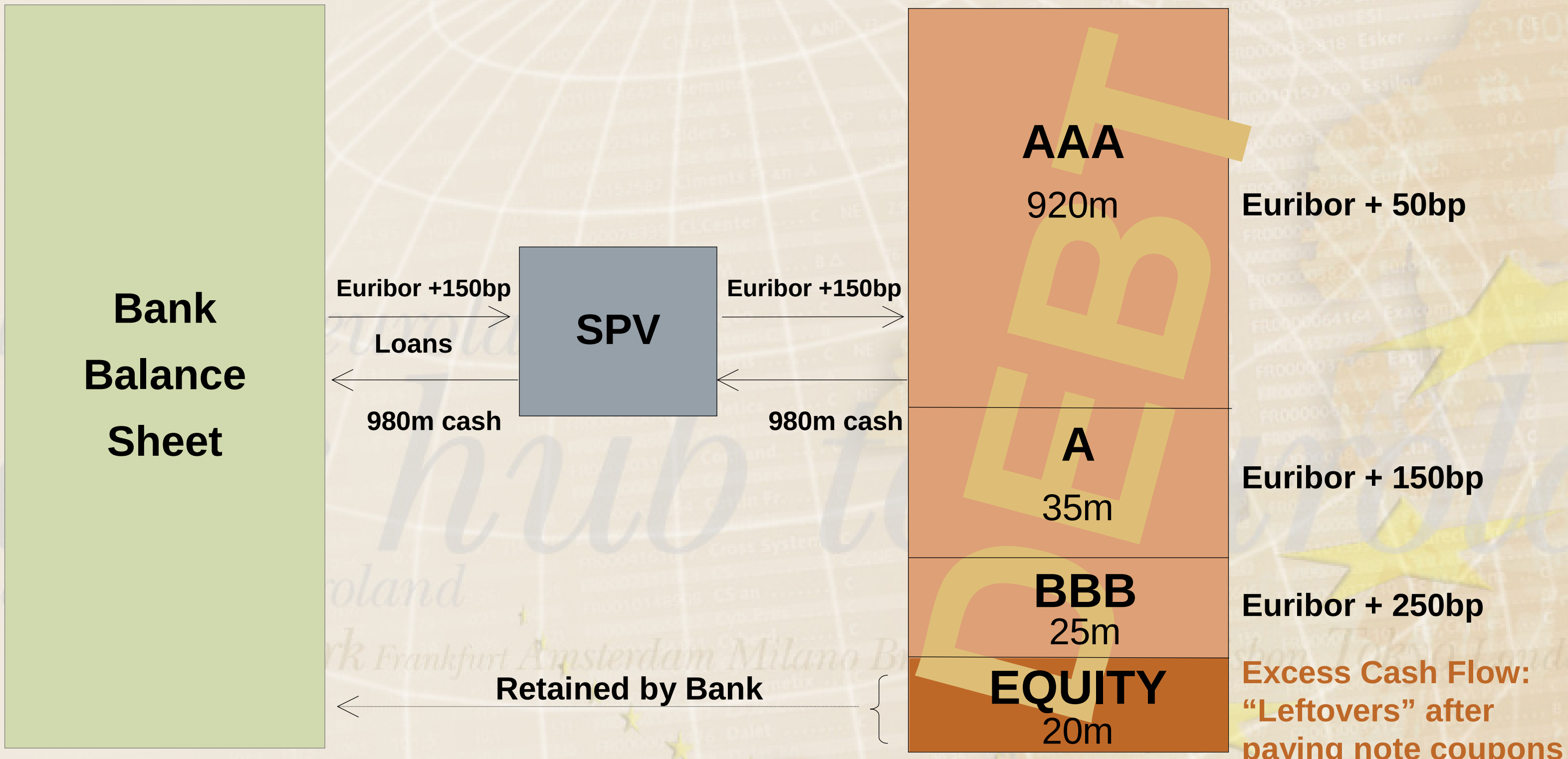
Quantitative research and the derivatives market : the case of structured derivatives

- With the search for higher yields and leverage and the standardization of the market for traditional derivatives, structured products increasingly occupy the front scene in the derivatives market.
- The design, analysis and management of structured derivatives calls for an even more active role of quantitative modelling
- For simpler derivative contracts, the design of the contract does not involve quantitative modelling: the pricing and risk management does.
- In the case of structured products, quantitative modelling already enters the picture when designing the payoffs.
- Thus, quantitative models play a more important role and financial engineers become involved in the life cycle of the products at a very early stage.

Impact of modeling approaches on the market: the case of Collateralized Debt Obligations (CDOs)

- The development of the market for Collateralized Debt Obligations, which exploded at the end of the 1990s, hinged on two elements:
  - The ability to attribute ratings to structured products such as single-tranche CDOs -> development of statistical models for analyzing the credit risk of large portfolios -> assessment of historical default rates
  - The ability to price the default risk/ exposure to default correlation of large portfolios of debt instruments -> development of first-generation CDO pricing models -> assessment of market-implied default rates
- *The subsequent surge of interest in the credit derivatives markets can be seen as resulting from discrepancy between historical and market-implied default rates*
  - *The systematic spread between market-implied and historical default rates has rendered yields on single tranche CDOs and other leveraged credit derivatives attractive for investors*
  - *This perception is based on quantitative indicators of default rates*

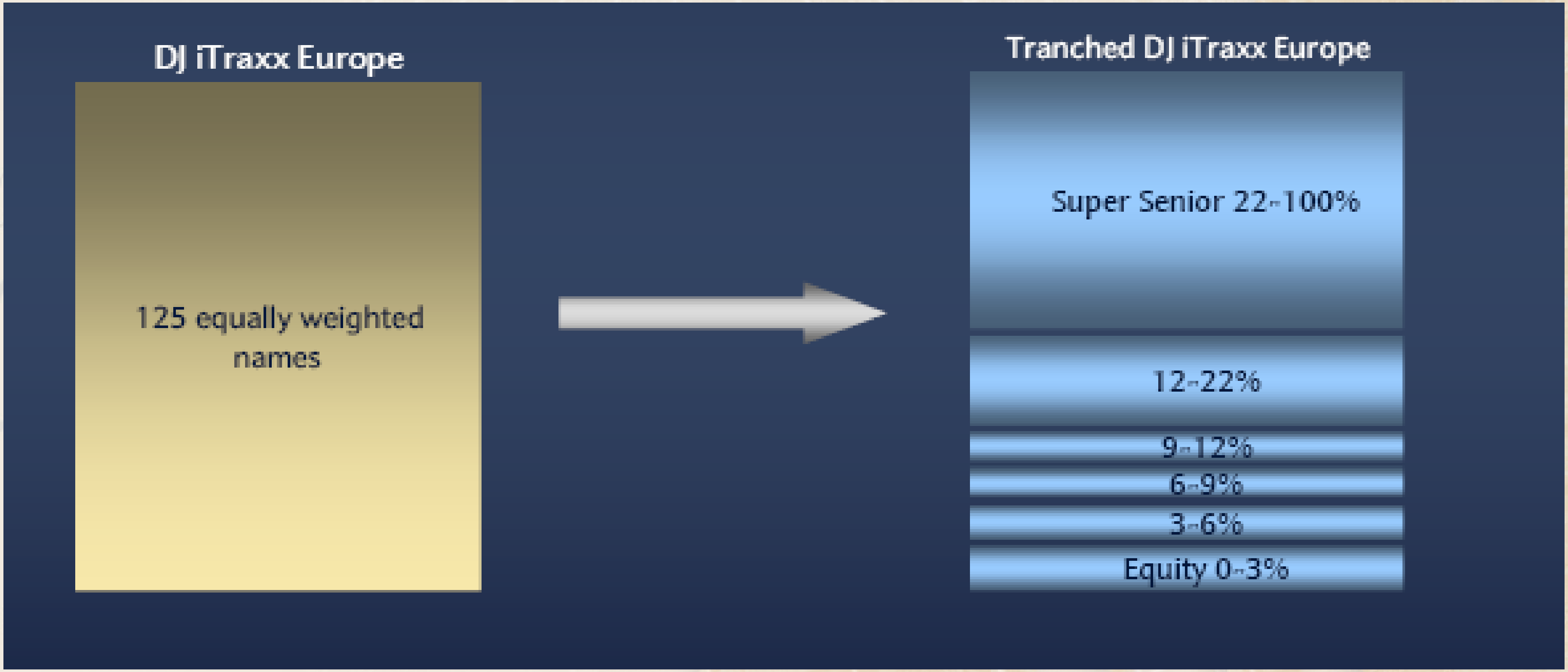
Collateralized debt obligation



Liability Cost:

$$\frac{(920m \times 50bp) + (35m \times 150bp) + (25m \times 250bp)}{980 \text{ million}} = 59bp$$

Credit indices: ITRAXX



PARIS EUROPLACE
FINANCIAL FORUM

CDO: cash flows

- l_t : Accumulated Loss at time t
- $l_t^{a,b}$: Tranche Loss at time t
- n : Number of names (companies)
- N_i : Notional
- R_i : Recovery (const/sto)
- τ_i : Default time
- a : Attachment point
- b : Detachment point

$$l_t = \sum_{i=1}^{n} \frac{N_i}{N}(1 - R_i) I_{\tau_i \leq t}$$

$$l_t^{a,b} = (l_t - a)_+ - (l_t - b)_+$$

$$\mathbb{E}\left[(l_t - a)_+\right] - \mathbb{E}\left[(l_t - b)_+\right]$$

$$C(t,k) = \mathbb{E}\left[(l_t - k)_+\right]$$

CDO market standard: Gaussian copula model

Gaussian copula model

Has become the market standard for quoting tranche spreads

- Transform default times to a Gaussian variable X_i via

$$N(X_i) = F_i(\tau_i) \iff X_i = N^{-1}(F_i(\tau_i))$$

- Model the dependence of the Gaussian variables X_i by a one-factor model

$$X_i = \sqrt{\rho}V + \sqrt{1-\rho}Z_i$$

where V and the Z_i are independent $N(0,1)$

Gaussian copula model : computational issues

Main object of interest: distribution of portfolio losses due to default

Two main approaches:

- Monte Carlo simulation
- Recursion methods (for homogeneous portfolios)

Both require a lot of computation: portfolio sizes are typically large (n=100-500)

It is of interest to obtain efficient approximations for calculating the loss distribution for large portfolios

Gauss & Poisson Approximation for portfolio loss distribution

Using **Stein's method** and **zero bias transformation**, we approximate $C(t,k)=\mathbb{E}\left[(l_t-k)_+\right]$

- Gauss approximation:

$$C(t,k)\simeq\underbrace{\int_{-\infty}^{+\infty}\mathrm{d}x\,\phi_{\sigma_W}(x)(x-\tilde{k})_+}_{\text{Bachelier's function}}+\frac{1}{6}\frac{1}{\sigma_W^2}\sum_{i=1}^{n}\mathbb{E}[X_i^3]\tilde{k}\phi_{\sigma_W}(\tilde{k})$$

Bachelier's function

where $\mathbb{E}[X_i^3]=\frac{(1-R)^3}{n^3}p_i(1-p_i)(1-2p_i)$

- Poisson approximation:

$$C(t,k)\simeq\mathcal{P}_{\lambda_V}(h)-\frac{1}{2}\Big(\sum_{i=1}^{n}\lambda_i^2\Big)\mathcal{P}_{\lambda_V}(\Delta^2 h)$$

where $\mathcal{P}_{\lambda_V}(\Delta^2 h)=\frac{1}{n}(1-R)\mathrm{e}^{-\lambda_V}\frac{\lambda_V^{m-1}}{(m-1)!}$

PARIS EUROPLACE

Gauss Approximation & Stochastic Recovery Rate (1/2)

We assume the recovery rate R_i follows a distribution whose first moments are:

$$\mu_{R_i} = \mathbb{E}\left[R_i\right] \qquad \sigma^2_{R_i} = \mathbb{E}\left[\left(R_i - \mu_{R_i}\right)^2\right] \qquad \gamma^3_{R_i} = \mathbb{E}\left[\left(R_i - \mu_{R_i}\right)^3\right]$$

We then have the same formulae as before except for a new term $\mathbb{E}[X_i^3]$

$$C(t,k) \simeq \underbrace{\int_{-\infty}^{+\infty} \mathrm{d}x\, \phi_{\sigma_W}(x)(x-\tilde{k})_+}_{\text{Bachelier's function}} + \frac{1}{6}\frac{1}{\sigma_W^2}\sum_{i=1}^{n}\mathbb{E}[X_i^3]\tilde{k}\phi_{\sigma_W}(\tilde{k})$$

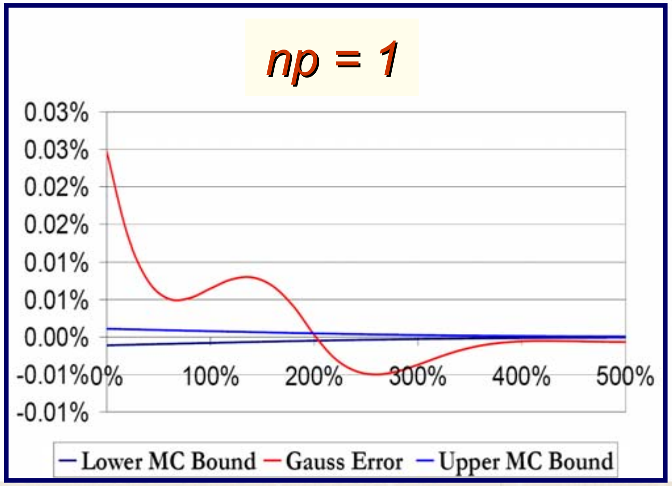
Bachelier's function

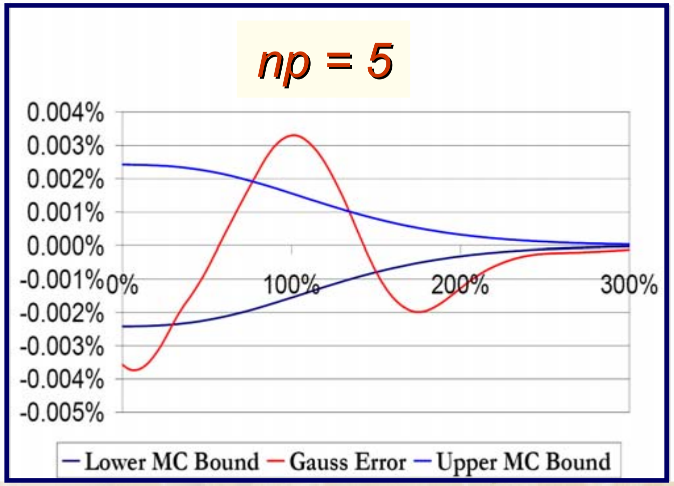
where

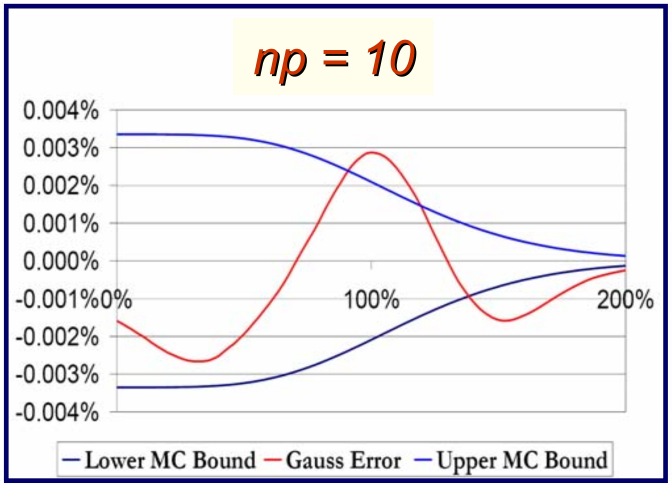
$$\mathbb{E}[X_i^3] = \frac{p_i}{n^3}\Big[(1-\mu_{R_i})^3(1-p_i)(1-2p_i) + 3(1-p_i)(1-\mu_{R_i})\sigma^2_{R_i} - \gamma^3_{R_i}\Big]$$

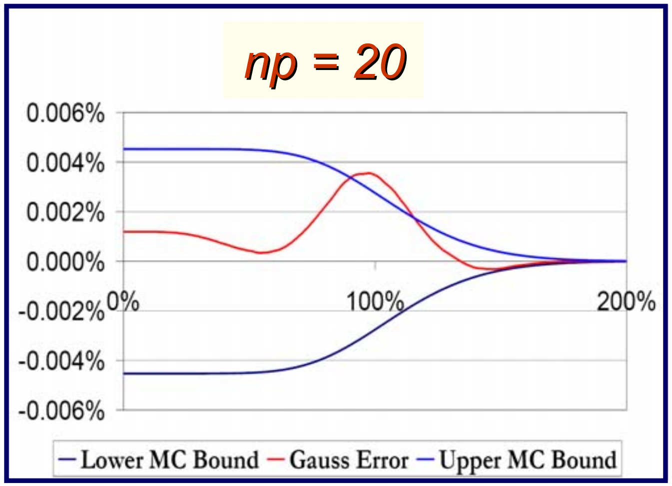
Gauss Approximation & Stochastic Recovery Rate (2/2)

Approximation Error versus strike normalized by the expected loss for different values of *np*.
The recovery rate follows a beta random variable (expectation = 50%, std dev = 26%)
We compare with a Monte Carlo method (1,000,000 simulations, 95% confidence interval)









Default correlation

Conditionally on U, the events $E_i = \{\tau_i \leq t\}$ are independent.
To completely specify a correlation model, one has to choose a function F such as:

$$\int_0^1 F(p,u)\mathrm{d}u = p, \quad 0 \leq F \leq 1$$

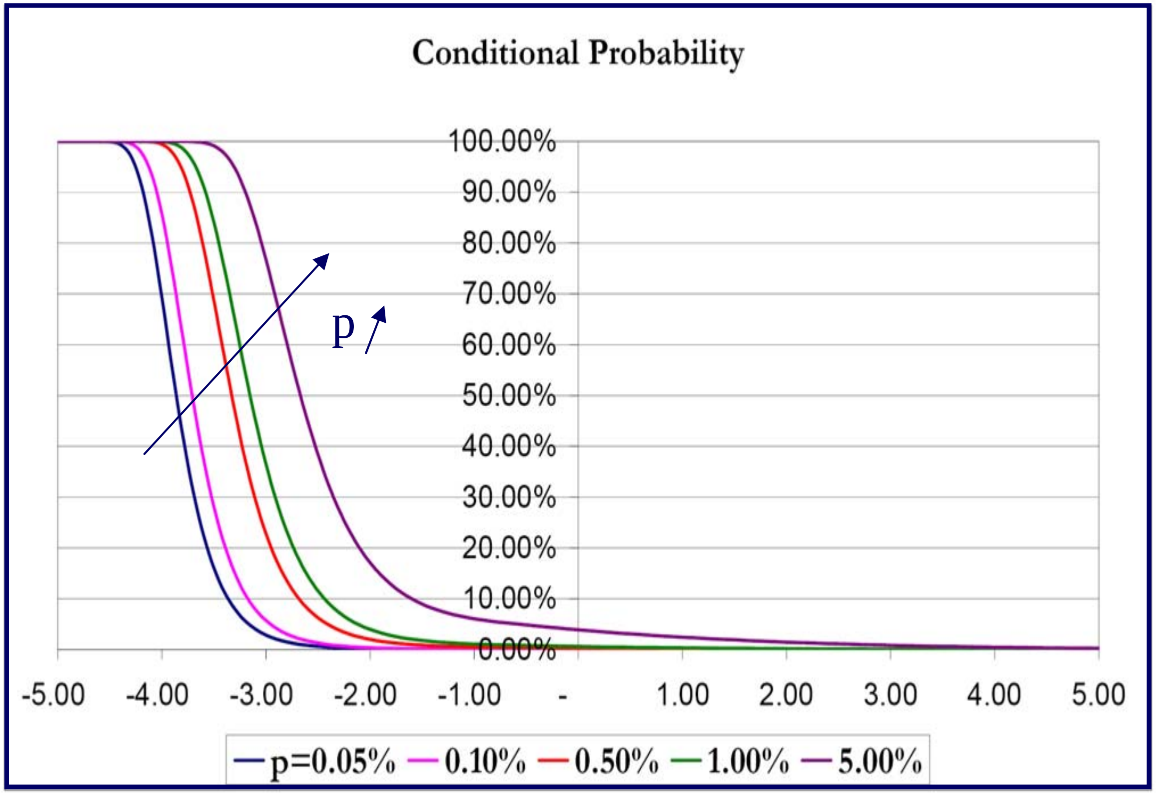
The standard Gaussian copula case with correlation ρ correspond to the function F :

$$F(p,u) = \mathcal{N}\left(\frac{\mathcal{N}^{-1}(p) - \sqrt{\rho}\mathcal{N}^{-1}(u)}{\sqrt{1-\rho}}\right)$$

Nevertheless, in the following tests, we apply a general approach where the function F is defined in a non parametric way in order to retrieve the observed market prices of tranches.

Conditional Probability (Normal factor)

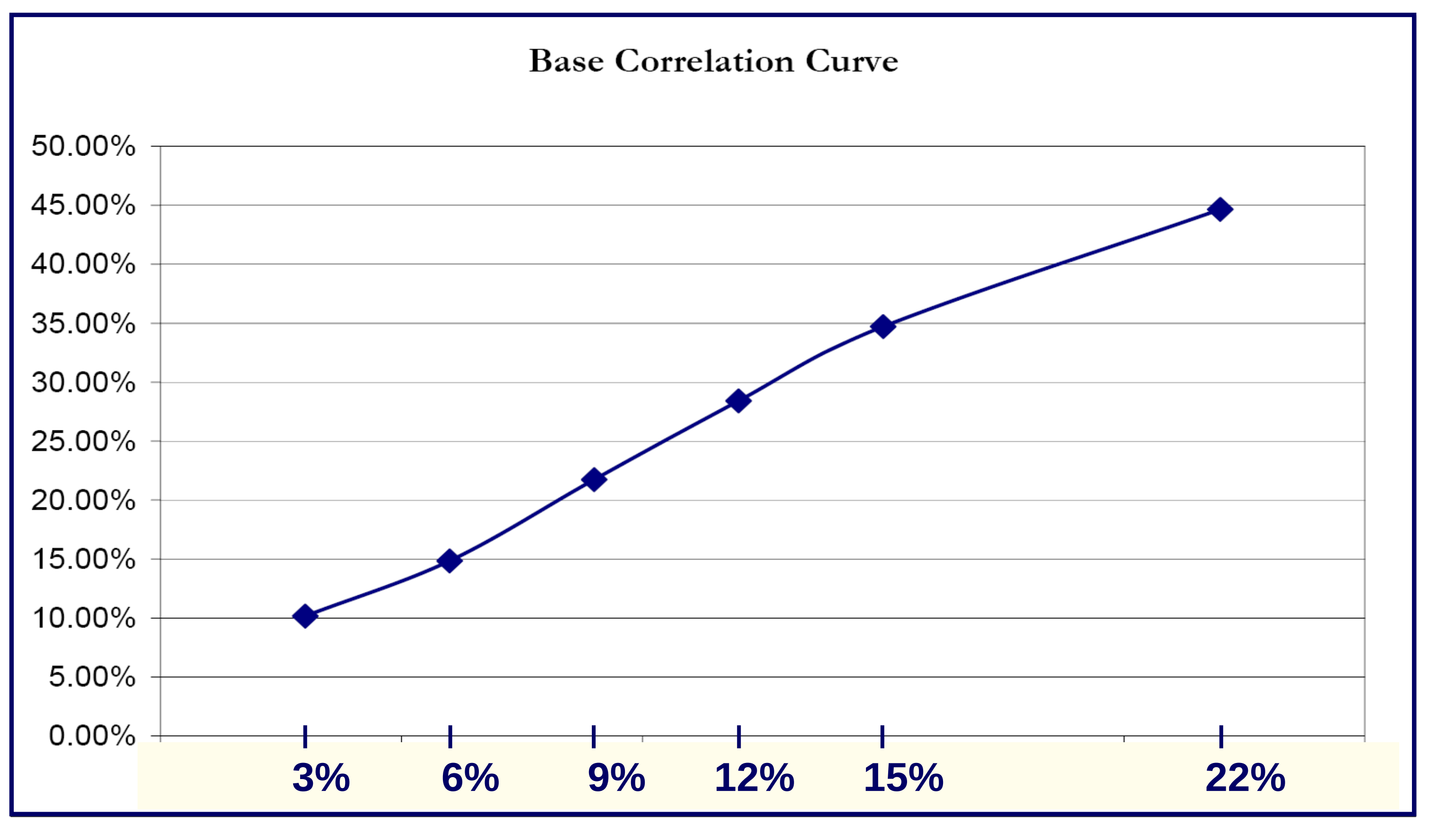
$$x \mapsto F(p, \mathcal{N}(x))$$



We calibrated on five years observed market prices of the tranches on a bespoke basket.
The base correlation will be …

Base correlation curve

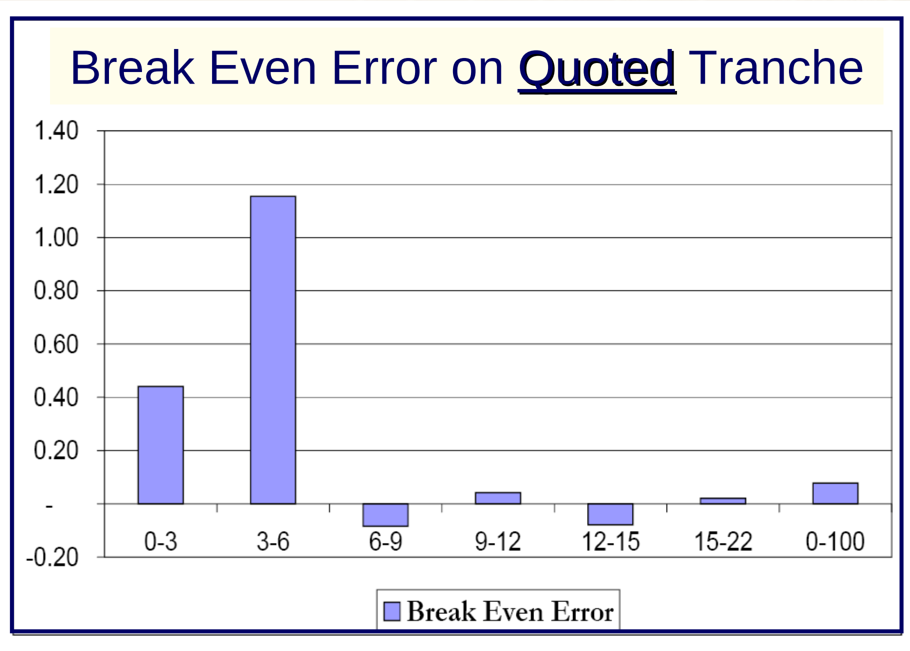
Data: CALYON.

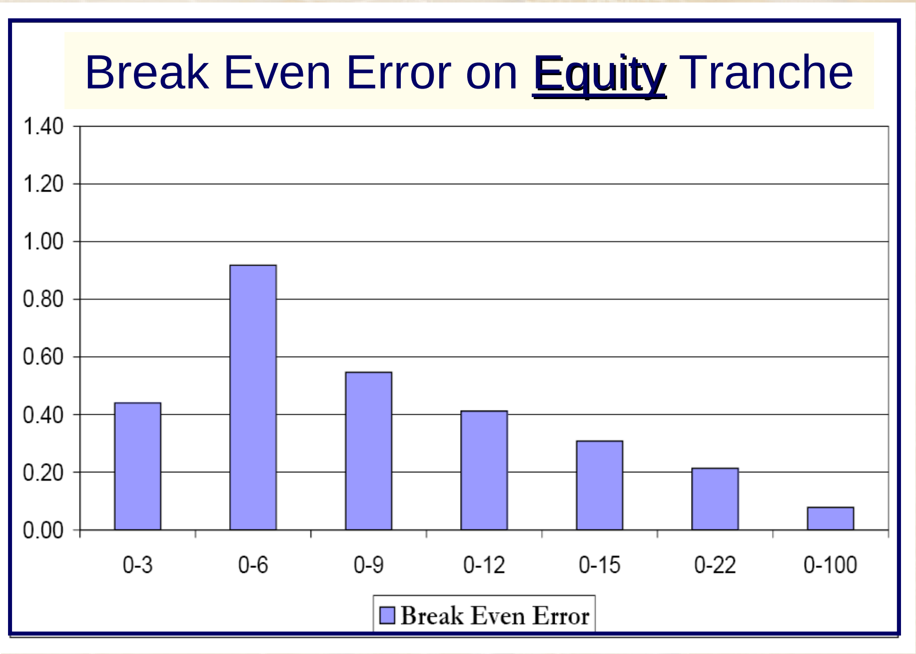


PARIS EUROPLACE FINANCIAL FORUM

Example

Comparison between the CDO pricing (Break Even) obtained from recursive method and the both approximations (Gauss & Poisson).





An efficient algorithm for pricing CDO tranches without simulation

✓ From the Stein's method and the Bias Transformation, we worked out two kinds of approximation (Gauss and Poisson) for the call function $C(t,k) = \mathbb{E}\left[(l_t - k)_+\right]$

✓ We showed the validity domain of both approximations is complementary. Indeed, if the sum of default probabilities (*np*) is greater than 15, then gauss approximation gives better results than Poisson approximation. Otherwise (*np* < 15), the Poisson approximation performs well.

✓Gaussian approximation in presence of a stochastic Recovery Rate

✓ We applied the both approximations for the CDO pricing. Max error = 1.2bp. We also studied the sensitivity to default probability.

Beyond the Gaussian copula: extracting implied default rates from market CDO prices

The Gaussian copula model is not able to match market quotes for CDOs

This observation has led to the development of alternative, more complex models for portfolio default risk

Calibration of these models to market data is a challenge

We have explored a class of “top-down” models where one represents the total portfolio loss due to default L(t)

The model is parameterized by the ‘transition rates’ of the loss: $a_n(t,T)$ denotes the probability of moving from n defaults to n+1 defaults.

First step: obtain $a_n(t,T)$ from market quotes of CDO tranches.

Beyond the Gaussian copula: extracting implied default rates from market CDO prices

Calibrate initial forward transition rates ($a_n(0,T)$) to market quotes for CDO tranches.

Two steps :

- from market tranche spreads to expected equity tranche loss

$$s(a,b) \longrightarrow C_t(T,a)$$

- from expected equity tranche loss to forward transition rates

$$C_t(T,a) \longrightarrow a_n(t,T)$$

where $C_t(T,a) = \mathbb{E}((a - L(T))^+ | \mathcal{F}_t)$

From market prices to expected tranche loss

- Step 1: extract expected tranche loss from market spreads of CDO tranches s(a,b)

Idea: derive a differential equation verified by the tranche spread.

(R Cont, I Savescu 2006)

$$s(a,b) = \frac{\int_t^T B(t,s)\mathbb{E}\left(dL_{a,b}(s) \middle| \mathcal{F}_t \right)}{\int_t^T B(t,s)\mathbb{E}\left(b - a - L_{a,b}(s) \middle| \mathcal{F}_t \right) ds}$$

$$\Downarrow$$

$$C_t(T,b) \quad = \quad C_t(T,a) + \left[(b - L(t))^+ - (a - L(t))^+\right] e^{-s(a,b)(T-t)}$$

$$s(a,b) \quad = \quad \frac{1}{T-t} \ln \frac{(b - L(t))^+ - (a - L(t))^+}{C_t(T,b) - C_t(T,a)}$$

From expected tranche loss to portfolio default rates

$$C_t(T,a) \longrightarrow a_n(t,T)$$

- Similar approach with the one proposed by B. Dupire for calculating local volatilities in a Black-Scholes model

- Forward Kolmogorov equation :

$$\frac{d}{dT}P(t,T) = P(t,T)A(t,T)$$

which can be written explicitly :

$$\begin{aligned} \frac{\partial}{\partial T}p_0(t,T) &= -a_0(T,t)p_0(t,T) \\ \frac{\partial}{\partial T}p_n(t,T) &= -a_n(T,t)p_n(t,T) + a_{n-1}(t,T)p_{n-1}(t,T) \end{aligned}$$

From expected tranche loss to portfolio default rates

A forward equation for equity tranche losses

$$\frac{dC(T,j)}{dT} = -a_{j-1}(T)C(,jT) - (a_{j-2}(T) - 2a_{j-1}(T))C(T,j-1) - \sum_{i=1}^{j-2}(a_{i-1}(T) - 2a_i(T) + a_{i+1}(T))C(T,i)$$

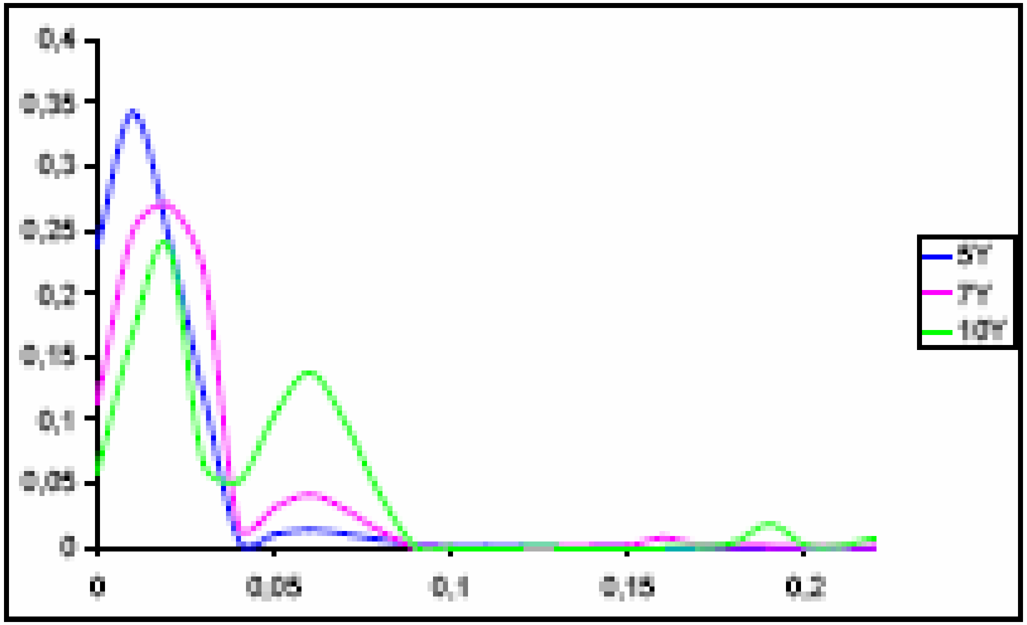
This differential equation allows to efficiently invert the market prices to obtain the portfolio default rates $\mathbf{a_n(t,T)}$, without using any complex numerical optimization procedure.

Empirical results for ITRAXX index (Data: SocGen)

Calibration results - ITRAXX 24 Apr 2006 (continuous and discrete cases)

	Attachment		Market	Computed spreads	
	Low	High	quotes	continuous	discrete
5Y	0 %	3 %	19.13%	19.18 %	19.133 %
	3 %	6 %	54 bp	52.93 bp	53.873 bp
	6 %	9 %	15.5 bp	15.73 bp	15.553 bp
	9 %	12 %	7.5 bp	7.501 bp	7.499 bp
7Y	0 %	3 %	37.5 %	37.487 %	37.504 %
	3 %	6 %	153 bp	153.06 bp	152.859 bp
	6 %	9 %	38 bp	38.08 bp	38.07 bp
	9 %	12 %	19.5 bp	19.48 bp	19.5 bp
10Y	0 %	3 %	48 %	47.98 %	47.99 %
	3 %	6 %	495.5 bp	495.414 bp	495.479 bp
	6 %	9 %	95 bp	95.22 bp	95.01 bp
	9 %	12 %	46 bp	45.89 bp	46.013 bp

Implied loss distributions for ITRAXX index: 5 year, 7 year and 10 years



PARIS EUROPLACE
FINANCIAL FORUM

In the search of higher yields: dynamic credit strategies

- Leveraged Super senior tranches
- Credit CPPI: constant proportion portfolio insurance

  DYNAMO (BNP Paribas – Credit agricole Asset management)

  Rozavel (JP Morgan - Credit agricole Asset management)
- CPDO: Constant proportion debt obligation

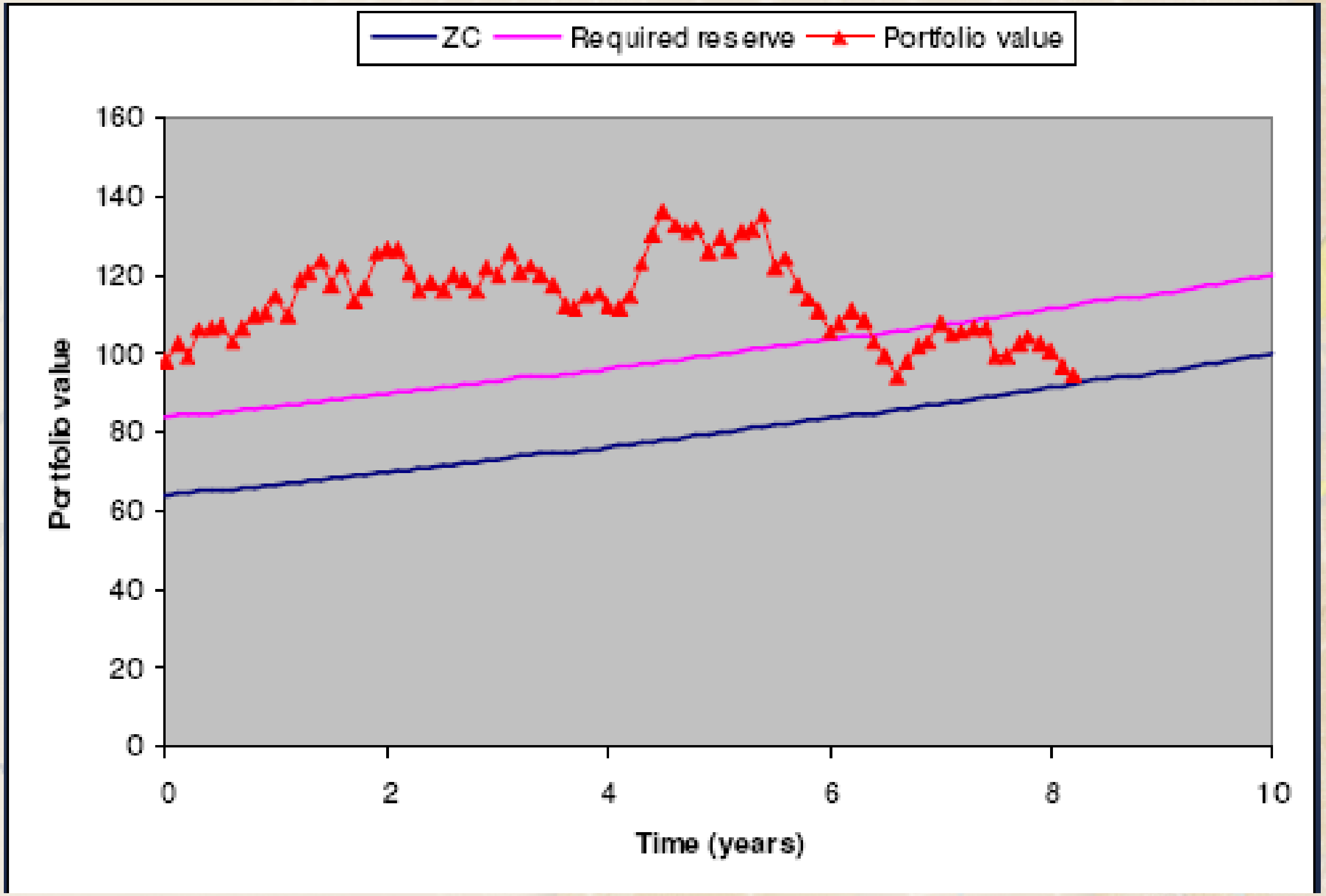
Constant proportion portfolio insurance (CPPI)

The term *portfolio insurance* refers to portfolio management techniques designed to guarantee that the portfolio value at maturity or up to maturity will be greater or equal to a given lower bound (floor), typically fixed as a percentage of the initial investment. These techniques allow the investor to limit downside risk while retaining some potential in case of an upside market move

The CPPI strategy is a self-financing strategy whose goal is to guarantee a fixed amount N of capital at maturity T. To achieve this, at any date t, a fraction of the portfolio is invested into the risky asset S_t and the remainder is invested into zero-coupon bond with maturity T and nominal N, whose price is denoted by B_t. Denoting the value of the fund by V_t,

- if $V_t > B_t$, the risky asset exposure (amount of money invested into the risky asset) is given by $mC_t \equiv m(V_t - B_t)$, where C_t is the 'cushion' and $m > 1$ is a constant multiplier.
- if $V_t \le B_t$, the entire portfolio is invested into the zero-coupon.

Constant proportion portfolio insurance (CPPI)



CPPI strategies and gap risk

- In the Black-Scholes model, whenever $\mu > r$, the expected return of a CPPI portfolio can be increased indefinitely and without risk, by taking a high enough multiplier.
- In reality, the possibility of reaching the floor, known as "gap risk", is widely recognized by CPPI managers: there is a nonzero probability that, during a sudden downside move, the fund manager will not have time to readjust the portfolio, which then goes crashing through the floor.
- Liquidity : many CPPI strategies are written on funds which may be thinly traded, leading to jumps in the market price due to liquidity effects.
- Since the volatility of the strategy is proportional to the leverage m, the risk of such loss increases with m, and in practice, the value of m should be fixed by relating it to an acceptance threshold for the probability of loss or some other risk measure.

Risk analysis of CPPI strategies

- Main risk: downward jumps of underlying portfolio due to defaults.
- We have proposed a simple framework for studying the "gap risk" of CPPI strategies, caused by downward jumps in the value of the underlying portfolio.
- Jump risk is not only significant for CPPI strategies but also leads to a criterion for adjusting the multiplier based on the investor's risk aversion.
- Our framework leads to analytically tractable expressions for the
  - probability of hitting the floor,
  - the expected loss and
  - the distribution of losses.
- Application: measuring "gap risk" for CPPI strategies

R Cont, P Tankov (2007) *Constant proportion portfolio insurance in presence of jumps in prices*.

Travaux en cours

- Role des criteres adoptes par les agences de notations dans le marche des CDOs
- Risque de modele dans les CDOs

Publications

- El Karoui, N., Y Jiao et D Kurtz: Gauss Poisson approximations for CDO loss distributions.
- R Cont, I Savescu (2006) *Extracting information on portfolio loss dynamics from CDO tranche spreads.* S&P Credit Risk Summit, Oct 2006
- R Cont, P Tankov (2007) *Constant proportion portfolio insurance in presence of jumps in prices.*
- R Bruyère, R Cont, C Jaeck, L Féry, T Spitz (2005) *Credit Derivatives*, Wiley Series in Financial Engineering.

En collaboration avec

- ✓ Ying JIAO (Ecole Polytechnique & ENSIMAG)
- ✓ David KURTZ
- ✓ Andreea MINCA (Ecole Polytechnique)
- ✓ Ioana SAVESCU (Ecole Polytechnique)
- ✓ Peter TANKOV (Universite de Paris VII)